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SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

Solutions of problems in No. 4, have been received as follows:

From Geo. L. Dake, 75 & 76; Prof. A. B. Evans, 75, 76, 77, 78, 79, 80, 81 & 82; E. S. Farrow, 75, 76 & 79; H. Heaton, 75, 76, 77, 78, 79, 80, 81 & 82; G. W. Hill, 81; Artemas Martin, 79 & 81; A. B. Nelson, 75, 79 & 81; L. Regan, 78; E. B. Seitz, 75, 77, 79 & 82; Walter Siverly, 75, 76, 77, 78, 79 & 82; Prof. C. M. Woodward, 76 & 77.

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75. "A stick of timber of uniform density and size from end to end, has a weight of 600 pounds suspended at one end, by which it is balanced horizontally on a fulcrum 6 feet from the end where the weight is suspended. If the fulcrum be 5 feet from the same end the stick will be balanced with a weight of 800 pounds. Required the weight and length of the stick."

SOLUTION BY CADET E. S. FARROW, WEST POINT, N. Y.

Since the stick is uniform, its centre of gravity is at its middle point. Let l equal its length, and w its weight (A and B representing the two ends, G its centre and C' and C the positions of the 1st and 2nd named fulcrums respectively). Taking moments about C and C' we have

$$w\times CG + 0\times CA = 600\times CB,$$

also $$w\times C'G + 0\times C'A = 800\times C'B;$$

or $$w\times(\tfrac{1}{2}l - 6) = 3600, \quad\text{and}\quad w\times(\tfrac{1}{2}l - 5) = 4000;$$

whence $$l = 30, \quad\text{and}\quad w = 400.$$

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76. "Within a given circle to draw three others tangent to it and touching each other externally, the sum of whose diameters shall be equal to that of the given circle."

77. "Within the circumference of a given circle, radius r, describe two equal circles, radii $\frac{2}{3}r$, which shall touch the first circle internally at opposite extremities of the diameter. Describe a fourth circle D, touching the first circle internally and each of the two equal circles externally. Describe a fifth circle E, a sixth circle F, &c., touching the first internally and one of the two equal circles and the circle D, E, F, &c., respectively, externally. Find an expression for the radii of the circles E, F, G, &c."

SOLUTION BY PROF. C. M. WOODWARD, ST. LOUIS, MO.

Preliminary Remarks. The properties of tangent circles used in the following solution are proved in an admirable article in the Mathematical Monthly, Vol. I. pp. 268 et seq., by Matthew Collins of Dublin. One of these properties may be stated thus:

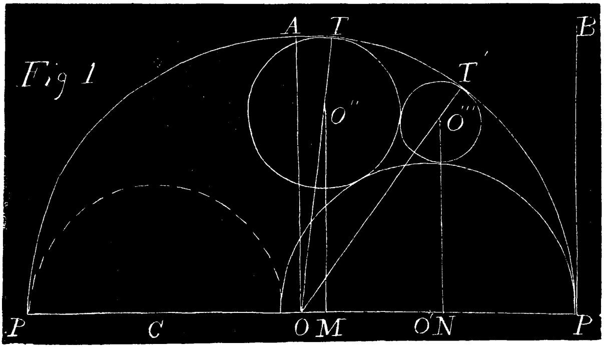
If a variable circle is tangent to two fixed circles, the distance of its centre from the radical axis of the fixed circles bears a constant ratio to its radius.

Another, a property of the "Arbelos," is older if not better known:

If O'' and O''' (Fig. 1.) *are any two circles tangent to each other and to O and O', and if the ratio*

$$\frac{O''M}{O''T} = n, \qquad \textit{then} \qquad \frac{O'''N}{O'''T'} = n + 2.$$

In the solution of problem 76, let the radius of the given circle $= 1$, and let the radii of the required circles, O', O'', O''', be r', r_2, r_3, and let fig. 1 be a general representation of the four circles.



By examining the special case of the dotted circle C, we see that

$$\frac{CP}{CP'} = \frac{MP}{O''T} = \frac{MP}{r_2} = \frac{NP}{r_3} = \frac{1 + r'}{1 - r'},$$

PB being the radical axis of O and O'; and, that if $O''M = nr_2$ we have $O'''N = (n + 2)r_3$. For brevity let

$$\frac{1 + r'}{1 - r'} = v, \text{ then} \begin{cases} O\ M = 1 - vr_2 \\ O''M = nr_2 \end{cases} \text{and} \begin{cases} O\ N = 1 - vr_3 \\ O'''N = (n + 2)r_3. \end{cases}$$

Hence

$$(1 - r_2)^2 = n^2r_2^{\,2} + (1 - vr_2)^2, \quad \ldots\ldots \quad (1)$$

$$(1 - r_3)^2 = (n + 2)^2r_3^{\,2} + (1 - vr_3)^2, \quad \ldots\ldots \quad (2)$$

$$1 = r' + r_2 + r_3. \quad \ldots\ldots \quad (3)$$

Equations (1) and (2) give

$$r_2 = \frac{2(v - 1)}{n^2 + v^2 - 1}, \ldots (4) \text{ and } r_3 = \frac{2(v - 1)}{(n + 2)^2 + v^2 - 1}. \ldots (5)$$

Since $1 - r' = \dfrac{2}{v + 1}$, the substitution of (4) and (5) in (3) gives

$$\frac{1}{n^2 + v^2 - 1} + \frac{1}{(n + 2)^2 + v^2 - 1} = \frac{1}{v^2 - 1}. \quad \ldots\ldots \quad (6)$$

Solving for n we get $\qquad n = \pm\, v - 1.$

Negative values of n indicate either that the ordinate of the centre of r_2 is negative, (i. e. O'' is in the other horn of the Arbelos), or that r_2 is itself negative, O'' being external to O. Hence, omitting the negative value of v, we have from (4) and (5)

$$r' = \frac{v-1}{v+1}, \quad r_2 = \frac{1}{v} = \frac{1-r'}{1+r'}, \quad r_3 = \frac{v-1}{v+1} \cdot \frac{1}{v} = r'r_2.$$

Moreover we have $OM = 1 - vr_2 = 0$,

$$ON = 1 - vr_3 = 1 - r' = OO',$$

$$O''M = nr_2 = \frac{v-1}{v} = (n+2)r_3 = O'''N.$$

These results show, when taken in connection with the remark as to the negative value of v;—

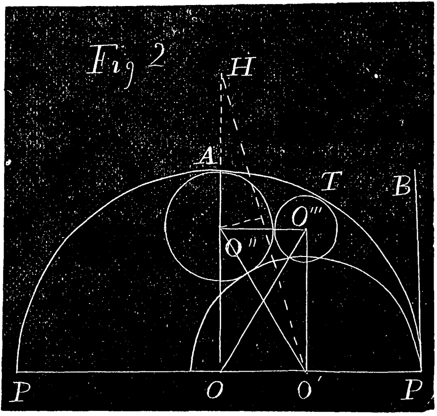
1°. That r' may be taken equal to any number, integral or fractional, positive or negative; as $a \div b$.

2°. That r_2 is given by a fraction whose numerator equals the denominator of r' *minus* its numerator; and whose denominator is the sum of the denominator and numerator of r'; as $(b - a) \div (b + a)$.

3°. That r_3 is the product of r' and r_2; as

$$\frac{a}{b} \cdot \frac{b-a}{b+a}.$$

4°. That *the four centres are always at the vertices of a rectangle.*



Sets of values of r', r_2, and r_3, can readily be given *ad infinitum.* Fig. 2 shows the circles in their correct position. The geometrical construction of any required set is indicated, AH being equal to $O'P$. Since $OO' = O''O''' = r_2 + r_3$ it is obvious that

$$r' + r_2 + r_3 = 1.$$

Cor. I. If $r' = \frac{2}{3}$ we have a ready solution of problem 77. In this case $v = 5$, $n = 4$ and, Fig. 3,

$$r_2 = \text{radius of } D = \tfrac{1}{5}$$

$$r_3 = \text{radius of } E = \tfrac{2}{15},$$

$$r_4 = \text{`` `` } F = \tfrac{2}{22} = \tfrac{1}{11},$$

. . . .

and obviously $$r_m = \frac{2(v-1)}{[n + 2(m-2)]^2 + v - 1} = \frac{2}{m^2+6}; \quad \ldots\ldots \quad (7)$$

hence $r_0 = \frac{1}{3}$, $r_1 = \frac{2}{7}$, $r_2 = \frac{1}{5}$, $r_3 = \frac{2}{15}$, $r_4 = \frac{1}{11}$, $r_5 = \frac{2}{31}$, $r_6 = \frac{1}{21}$ &c.

Cor. II. Since $v = (1 + r') \div (1 - r')$ and $n = 2r' \div (1 - r')$, (7) gives

$$r_m = \frac{r'(1-r')}{[(3-m)r' + m - 2]^2 + r'}, \quad \ldots\ldots\ldots \quad (8)$$

which gives in terms of r' (which may be any number) the radius of any circle in a series, the centre of r_2 being on the radius $OA \perp$ to PP'.

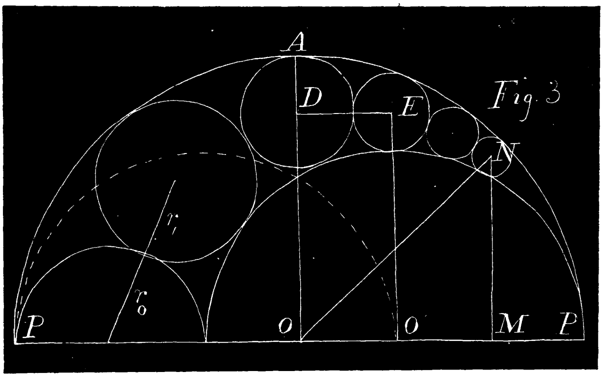


Fig. 3

Cor. III. If $n = 2r' \div (1 - r') = v - 1$ is an even number, repeated subtractions of 2 must reduce it to zero, so that the largest circle of the series has its centre on PP', and the series may be called *symmetrical.* This is the case when v is an odd integer, as in Fig. 3.

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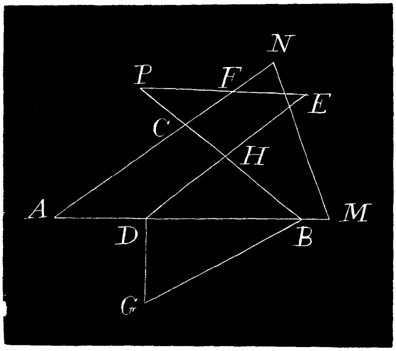
78. "From a given point without a triangle to draw a line which bisects the area of the triangle."

SOLUTION BY PROF. A. B. EVANS, LOCKPORT, N. Y.

Let NAM be the given triangle and P the given point. Draw PE parallel to AM and construct the parallelogram $AFED$ equal to one half the area of the triangle NAM. Draw DG perpendicular to AM and equal to PF, and GB equal to PE.



From the similar triangles PHE, PCF, BDH, since $PE^2 - PF^2 = DB^2$

$PHE - PCF = BDH$,

$\therefore \quad ACB = AFED = \frac{1}{2}NAM.$

Note. In Chauvenet's Geometry, exercise 249, p. 323, we find the following question which is more general than No. 78, and which may be solved by making the parallelogram $AFED$ equal to the part to be cut off from the angle NAM by a line through P:

"Through a given point, draw a straight line which shall form with two given intersecting straight lines a triangle of a given area."

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79. "Prove that the formula $2^{n-1}(2^n - 1)$ represents a perfect number when $2^n - 1$ is prime."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

A number which is equal to the sum of all its divisors 1 inclusive, is called a perfect number.

When $2^n - 1$ is prime the sum of all the divisors of $2^{n-1}(2^n - 1)$ including the number itself

$$= (1 + 2 + 2^2 + \ldots\ldots + 2^{n-1})(1 + 2^n - 1) = 2^n(2^n - 1),$$

which is twice the number. Therefore $2^{n-1}(2^n - 1)$ is a perfect number when $2^n - 1$ is prime.

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80. "The relation

$$\underline{|n} = 1 + A_1\frac{n}{\underline{|1}} + A_2\frac{n(n-1)}{\underline{|2}} + A_3\frac{n(n-1)(n-2)}{\underline{|3}} + \ldots + A_n\frac{\underline{|n}}{\underline{|n}}$$

is true for all positive integral values of n; show that

$$\frac{A_r}{\underline{|r}} = 1 - \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} \ldots\ldots + \frac{(-1)^r}{\underline{|r}},$$

where r is an integer less than n."

[There was a slight misprint in the announcement of this question. "*Ar*" should have been A_r.]

SOLUTION BY H. HEATON, DES MOINES, IOWA.

Put $A_r = x^r = [(x+1) - 1]^r = (x+1)_r$

$$- \frac{r}{\underline{|1}}\left(x+1\right)^{r-1} + \frac{r(r-1)}{\underline{|2}}\left(x+1\right)^{r-2} \ldots + (-1)^r$$

Then $A_r = \left(A_r + \frac{r}{\underline{|1}}A_{r-1} + \frac{r(r-1)}{\underline{|2}}A_{r-2} \ldots + 1\right) - \frac{r}{\underline{|1}}\left(A_{r-1} + \frac{r-1}{\underline{|1}} \times A_{r-2} \ldots + 1\right) + \frac{r(r-1)}{\underline{|2}}\left(A_{r-2} + \frac{r-2}{\underline{|1}}A_{r-3} + \ldots + 1\right) \ldots\ldots \left(-1\right)^r$.

But from the conditions of the problem it is easily seen that the quantity in the first parenthesis $= \underline{|r}$, that in the second $= \underline{|r-1}$ that in the third $= \underline{|r-2}$ &c. Hence

$$A_r = \underline{|r} - \frac{\underline{|r}}{\underline{|1}} + \frac{\underline{|r}}{\underline{|2}} - \frac{\underline{|r}}{\underline{|3}} \ldots + \frac{(-)^r\underline{|r}}{\underline{|r}},$$

$$\therefore \quad \frac{A_r}{\underline{|r}} = 1 - \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} \ldots + \frac{(-1)^r}{\underline{|r}}.$$

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81. "Prove that, *identically*,

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} \ldots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{2n}."$$

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

$$\text{Let}\quad P = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n-1} \dots\dots\dots\dots \quad (1)$$

$$\text{and}\quad Q = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots + \frac{1}{2n} \dots\dots\dots\dots \quad (2)$$

$$\text{By subtraction,}\quad P - Q = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{2n} \dots\dots \quad (3)$$

$$\text{By addition,}\quad P + Q = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2n} \dots\dots \quad (4)$$

Subtracting twice (2) from (4),

$$P - Q = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$$

82. "If the parabolic orbits of two comets intersect the circular orbit of the earth in the same two points, then if t_1 and t_2 be the times in which the comets move from one point to the other, $(t_1 + t_2)^{2/3} + (t_1 - t_2)^{2/3} = \left(\frac{4}{3\pi}\right)^{2/3}$: a year being the unit of time."

SOLUTION BY WALTER SIVERLY, OIL CITY, PA.

Let c represent the length of the chord of the earth's orbit joining the two points. It is shown in works on Central Forces that,

$$t_1 = \frac{1}{12\pi}\left(2 + c\right)^{\frac{3}{2}} + \left(2 - c\right)^{\frac{3}{2}},$$

$$t_2 = \frac{1}{12\pi}\left(2 + c\right)^{\frac{3}{2}} - \left(2 - c\right)^{\frac{3}{2}}.$$

Adding and subtracting,

$$t_1 + t_2 = \frac{1}{6\pi}\left(2 + c\right)^{\frac{3}{2}}, \qquad t_1 - t_2 = \frac{1}{6\pi}\left(2 - c\right)^{\frac{3}{2}}.$$

Whence

$$\left(t_1 + t_2\right)^{\frac{2}{3}} = \left(2 + c\right)\left(\frac{1}{6\pi}\right)^{\frac{2}{3}},$$

$$\left(t_1 - t_2\right)^{\frac{2}{3}} = \left(2 - c\right)\left(\frac{1}{6\pi}\right)^{\frac{2}{3}}.$$

Eliminating c,

$$\left(t_1 + t_2\right)^{\frac{2}{3}} + \left(t_1 - t_2\right)^{\frac{2}{3}} = \left(\frac{4}{3\pi}\right)^{\frac{2}{3}}.$$

PARADOX. BY G. SHAW, KEMBLE, ONT., CANADA.—Suppose $a = b$. Multiply both sides of this equality by b, and we have $ab = b^2$. Subtract a^2 from each side of this equation and we have $ab - a^2 = b^2 - a^2$. . . (1) By factoring (1) we have $(b - a)a = (b - a)(b + a)$. Divide this eqn. by $b - a$ and we have $a = b + a$ or, because $b = a$,

$$a = a + a = 2a.$$

Dividing by a we have $1 = 2$.

[The fallacy in the above obviously consists in considering a and b as *like* and *unlike* at the same time. For, in the first member of (1) because $a=b$, we have a zero factor, $b - a = 0$, and in the second member we have the same zero factor, $b - a$. Now when a and b are unlike the only two factors of $b^2 - a^2$ are $a - b$ and $a + b$, neither of which can be zero, but when $b = a$ then $b^2 - a^2 = a^2 - a^2$, the only two factors of which are $a - a$ and a; the first of these two factors, $a - a$, being zero, the other factor a, may be repeated as often as we please without changing the value of the product.—Ed]

PROBLEMS.

83. BY GEO. L. DAKE, CLEVELAND, OHIO.—A point is given within two lines which form a given angle with one another. Required the shortest line which can be drawn through this point, terminated by the given lines.

84. BY PHILLIP HOGLAN, NEWCOMERSTOWN, OHIO.—The centres of two spheres whose radii are 12 ft. and 5 ft., respectively, are at opposite extremities of the diameter of a circle of 13 ft. radius. Find a point in the cirumference of this circle from which the greatest portion of spherical surface is visible.

85. BY PROF. JAMES G. CLARK, LIBERTY, MO. — In a quadrilateral there are given, the length and position of the lower base, the lengths of the two sides, the length of the upper base and the position of a point through which it passes: required to construct the quadrilateral.

86. BY PROF. J. S. HAYES, HODGENVILLE, KY.— Prove that the attraction of a sphere of uniform density upon an external point is the same as if all the matter of the sphere were concentrated at its centre.

87. BY G. M. DAY, LOCKPORT, N. Y.— There are n tickets in a bag numbered 1, 2, 3 . . . n. A man draws three tickets together at random and is to receive a number of shillings equal to the product of the numbers he draws. Find the value of his expectation.